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Uncertainty Product for a Subensemble of Particles

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Abstract

The Heisenberg uncertainty principle derives from an assumption that all physically permissible location values x are included within the ensemble of observations. By contrast, we consider here a case where observation is made over a finite subinterval of the permissible x values. The resulting uncertainty product for x and k_x (wave momentum) is then inconsistent with the Heisenberg principle, as might be expected.

The Heisenberg uncertainty principle, by its derivation and usual interpretation, applies to an ensemble of measurements of position x extending over *all* space, $|x| \leqslant \infty$. However, there may be circumstances where an experimenter is interested in measuring x over only a particular subinterval of its possible domain. In this circumstance, because a basic premise of the principle is unsatisfied, it might seem reasonable to expect an uncertainty product *less* than the Heisenberg value of one-half. We intend to describe an experiment for which this is the case.

The situation is illustrated in Figure 1. Electrons individually enter the aperture of length L and head toward a photographic emulsion on screen S. The intervening space is free space. All electrons are constrained, before entering the aperture, to have an upper bound K_0 to their wave momentum (as obtained, for example, by use of a magnetic velocity sorter[2]). Then L must obey

$$K_0 L > \tfrac{1}{2} \tag{1}$$

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[2] The Dempster or Bainbridge mass spectrograph, for example, contains such a device.

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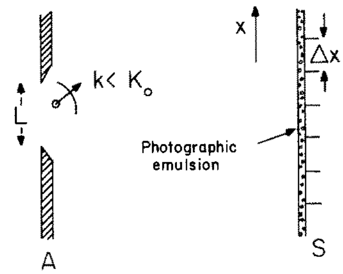

Figure 1—Individual electrons, obeying $k \leqslant K_0$, enter through the aperture of length L and traverse free space toward a photographic emulsion on screen S. An ensemble of detections of position x on S is built up by developing the emulsion and replacing it with another after each single electron traversal. We are concerned with the spreads in x and in k_x for those detections that fall within a *single* subinterval Δx of the screen S.

in order for the electrons to (1) satisfy the Heisenberg principle within the aperture and (2) pass through the aperture.

Screen S is covered by a continuous photographic emulsion that is physically subdivided into contiguous intervals of length Δx by fiducial scratch marks. Each electron is allowed to traverse the space from A to S and register, by means of a developable photographic grain, on S. After each electron traversal, the photographic emulsion is removed from S, developed, and replaced with a new photographic emulsion having intervals Δx marked off as before and in registration (ideally) with them. In this manner, particle events falling within some Δx of the *screen* S may be identified.

We now consider the spread in x and k_x for electrons that are detected, in this manner, within any one interval Δx of S. The spread σ_x in x is no greater than Δx (or it equals $\Delta x/\sqrt{6}$ if all values of x on the interval are equally likely).

The spread in k, i.e., σ_k, is $(\langle k_x{}^2\rangle - \langle k_x\rangle^2)^{1/2}$. From this, $\sigma_k < \langle k_x{}^2\rangle^{1/2}$. But every electron has been constrained to obey $k < K_0$ as a precondition to passage through the aperature L. Hence, it is impossible for a single electron to strike S with a k_x exceeding K_0, and therefore $\langle k_x{}^2\rangle < K_0{}^2$ over the ensemble. (Alternatively, this inequality results from conservation of *expected energy* from A to S over the entire ensemble of events within Δx.) Hence

$$\sigma_k < K_0 \tag{2}$$

At this point we have shown that

$$\sigma_x \sigma_k < \Delta x K_0 \tag{3}$$

Now Δx is merely the size of the interval over which we observe developed grains, i.e., register arrivals. Suppose we choose to observe over an interval

$$\Delta x = 1/(3K_0) \tag{4}$$

(Note that we may choose to make K_0 small so that this Δx is actually of macroscopic dimensions, say 1 cm if we want.) Then equation (3) becomes

$$\sigma_x \sigma_k < 1/3 \tag{5}$$

This is the uncertainty product over the particular ensemble of events we choose to observe. Although a very small fraction of particles that enter aperture L will be detected in our interval, Δx, if we allow a sufficiently large number of particles to enter L, then our ensemble can be made arbitrarily large.

Conversely, if *all* events over the entire screen S are counted, the resulting uncertainty product will obey the Heisenberg value of one-half. It is only because we choose to observe a small subensemble of the totality of events that we can have the result (5). However, since even this subensemble can comprise an arbitrarily large number of events, the result (5) at least violates the "spirit" of the principle.

In rebuttal, it might be stated that electrons will not choose to strike the emulsion *anywhere* [just as an electron of momentum $k_x < K_0$ will refuse to pass through a slit whose length is smaller than $1/(2K_0)$ so as not to violate the Heisenberg principle]. However, an electron traveling toward S cannot anticipate that it will be measured *after* it strikes the emulsion there. Furthermore, it does not see an array of slits of opening Δx but rather a continuous (except for the scratch marks) photographic emulsion. Therefore, it will strike it.

Alternatively, it might be argued that there is an inherent uncertainty in locating *where* on S the particular interval Δx lies because of uncertainty in positioning the individual emulsions. And this uncertainty, when added to Δx, will permit the Heisenberg principle to be satisfied. However, as mentioned above, Δx need not be very small to satisfy condition (4). By making K_0 sufficiently small, we can have Δx the order of a centimeter. The uncertainty in locating the position of Δx can now be made at least three orders of magnitude smaller than Δx. When this additional uncertainty is added to Δx and multiplied by K_0 (the uncertainty in k_x) the product must be very close to 1/3, i.e.,

$$\Delta x(1 \pm 10^{-3})K_0 = \Delta x(1 \pm 10^{-3})1/(3\Delta x)$$

by equation (4),

$$= 1/3 \pm 1/3 \times 10^{-3} < \tfrac{1}{2}$$

Finally, we must consider the effect of the scratchmarks on an impinging electron. Can the scratchmarks make the emulsion appear to the electron as a series of slit jaws so that each Δx interval resembles a slit opening through which the electron cannot pass because of the small opening Δx in equation (4)? For this to be so, the scratches would have to behave like infinite, impenetrable potentials (Schiff, 1955). There is nothing in the makeup of a scratch in an emulsion that would suggest such behavior.

Reference

Schiff, L. I. (1955). *Quantum Mechanics* (McGraw-Hill, New York), p. 34.